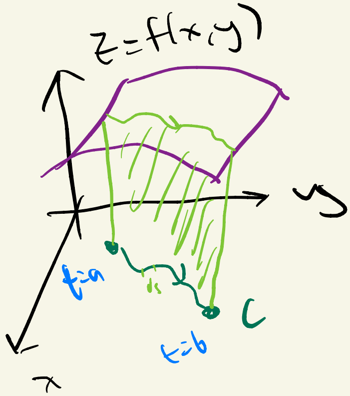
8-14

Last time...

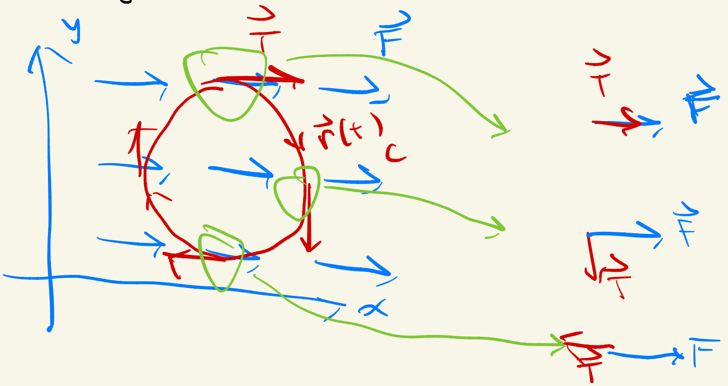


- Line integrals allow us to integrate scalar functions $f(x,y,z)$ over 1D (oriented) curves in 3D space:

$$\boxed{\int_C f(x,y,z)\,ds = \int_a^b f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt}$$

choose parametrization $\vec{r}(t)$, $a \le t \le b$

- Vector fields $\vec{F}(x,y,z) = M(x,y,z)\hat{i} + N(x,y,z)\hat{j} + P(x,y,z)\hat{k}$ assign a vector to each coordinate (x,y,z)



Use dot product at every point (x,y,z) to get scalar function $(\vec{F}\cdot\vec{T})(x,y,z)$ that we can integrate

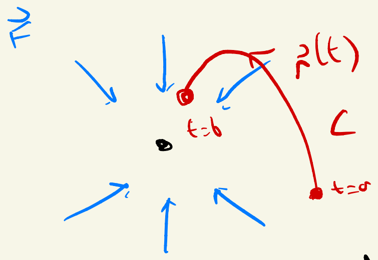
Line integral of $\vec{F}$ over C:

$$\boxed{\int_C \vec{F}\cdot\vec{T}\,ds = \int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt}$$

choose param $r(t)$, $a \le t \le b$

These have physical meaning:

- If $\vec{F}$ is a force field, like gravity or electric field, then



$$\int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot \vec{r}\,'(t)\,dt$$

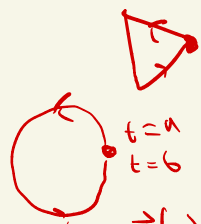
Is work done by a field $\vec{F}$ on object tracing oriented path C

- If $\vec{F}$ is a velocity field (like the swimmer in the ocean example), then

$\vec{F}$

$$\int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot \vec{r}\,'(t)\,dt$$

Is the flow of $\vec{F}$ along C.



$\vec{r}(a)=\vec{r}(b)$

Special case:
if C is a closed curve

Circulation $= \oint_C \vec{F}\cdot d\vec{r}$

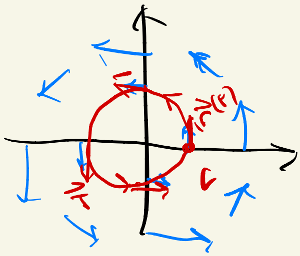
this just means closed curve

curves should be "nice" = simple, closed, smooth (later)

Ex: Circulation and Flow integrals

Find the circulation of the field $\vec{F} = -y\hat{i} + x\hat{j}$ around the circle $\vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j}$, $0 \le t \le 2\pi$

Soln:



on circle:

want to calculate

$$\oint_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt$$

$$\vec{F}(\vec{r}(t)) = -y\hat{i} + x\hat{j}$$
$$= -\sin t\,\hat{i} + \cos t\,\hat{j}$$

$$\vec{r}\,'(t) = (-\sin t)\hat{i} + (\cos t)\hat{j}$$

$$\Rightarrow \vec{F}\cdot\frac{d\vec{r}}{dt} = (-\sin t\,\hat{i} + \cos t\,\hat{j})\cdot(-\sin t\,\hat{i} + \cos t\,\hat{j})$$
$$= \sin^2 t + \cos^2 t$$
$$= 1$$

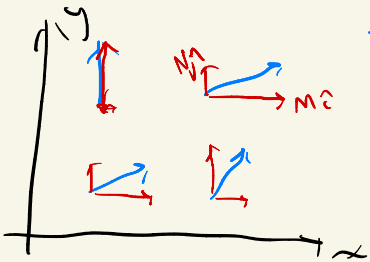
$$\Rightarrow \text{circulation} = \oint_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} 1\,dt$$
$$= 2\pi$$

"Circulation detects amount of "rotation" in vector field"

We'll learn about another type of vector field integral called a flux integral in a moment.

We'll first discuss a useful notation called "differential forms".

These allow us to study individual components of a vector field:



$\vec{F} = M\hat{i} + N\hat{j}$

what if we only want to measure one component?

e.g. only measure M?

$\vec{F}(x,y,z) = M(x,y,z)\,\hat{i} + N(x,y,z)\,\hat{j} + P(x,y,z)\,\hat{k}$

$\vec{r}(t) = g(t)\,\hat{i} + h(t)\,\hat{j} + k(t)\,\hat{k}$

write

$$\int_C \vec{F}\cdot\frac{d\vec{r}}{dt} = \int_C \left(M\hat{i} + N\hat{j} + P\hat{k}\right)\cdot\left(g'(t)\hat{i} + h'(t)\hat{j} + k'(t)\hat{k}\right)dt$$

$$= \int_a^b M(\vec{r}(t))\,\underbrace{g'(t)\,dt}_{\text{"little piece of length in x direction"}} + \int_a^b N(\vec{r}(t))\,h'(t)\,dt + \int_a^b P(\vec{r}(t))\,k'(t)\,dt$$

$$\overset{\text{def}}{=} \int_C M\,dx + \int_C N\,dy + \int_C P\,dz$$

only measures the vector field's $\hat{i}$ direction

Idea:

$\vec{r}(t) = g(t)\,\hat{i} + h(t)\,\hat{j}$

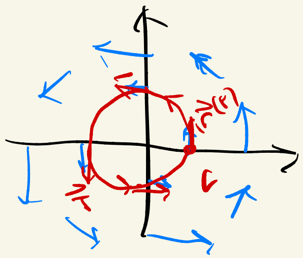
$x(t) = g(t)$

$dx = g'(t)\,dt$

Ex: Circulation (with forms!)

Find the circulation of the field $\vec{F} = -y\hat{i} + x\hat{j}$ around the circle $\vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j}$, $0 \le t \le 2\pi$

Soln:



$$M(x,y) = -y$$

$$N(x,y) = x$$

$$\vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j}$$

$$\Rightarrow \quad x(t) = \cos t$$

$$dx = -\sin t\,dt$$

$$y(t) = \sin t$$

$$dy = \cos t\,dt$$

$$\Rightarrow \int_C \vec{F} \cdot d\vec{r} = \int_C M\,dx + \int_C N\,dy$$

$$= \int_0^{2\pi} -y(-\sin t\,dt) + \int_0^{2\pi} x(\cos t\,dt)$$

$$= \int_0^{2\pi} -\sin t(-\sin t\,dt) + \int_0^{2\pi} \cos t(\cos t\,dt)$$

$$= \int_0^{2\pi} 1\,dt$$

$$= 2\pi$$

perspective will be useful for calculating flux integrals.

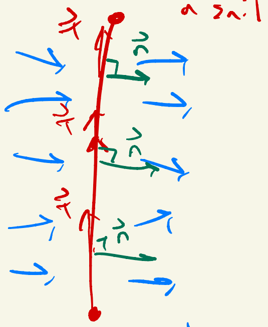
Work, Flow, ... we have another vector field integral

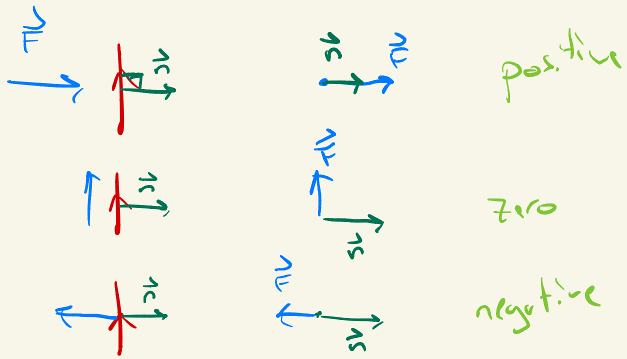
Flux integral

Idea of flux:

$\vec{F}$ = wind velocity field

How much does the sail feel from the wind?





Def:

If C is an oriented, simple, smooth curve in domain of a continuous vector field $\vec{F} = M(x,y)\hat{i} + N(x,y)\hat{j}$

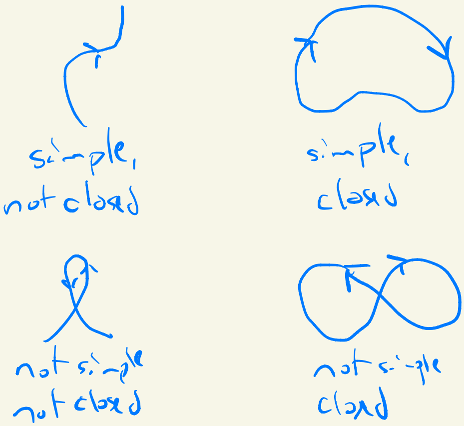
The flux across C is

$$\text{Flux} = \int_C \vec{F} \cdot \vec{n} \, ds = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{n} \, |\vec{r}\,'(t)| \, dt$$

choose parametrization $\vec{r}(t)$

Three things:

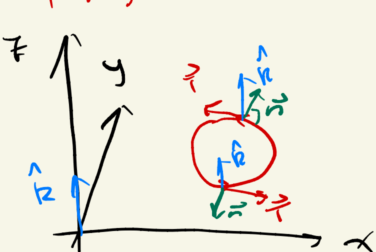
- A curve is simple if it does not intersect itself
- Important case for flux: closed curves (loops) which return to their start ($\vec{r}(a) = \vec{r}(b)$)



Integrals over closed curves:

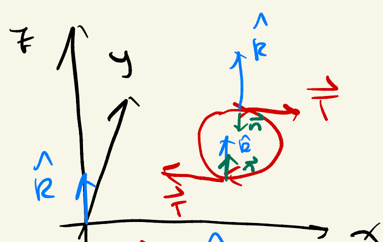
$$\oint (\ )\, ds$$

- Where does $\vec{n}$ point? decided by orientation of C + right hand rule



$$\vec{T} \times \hat{k} = \vec{n}$$

counterclockwise CCW = outward normal



$$\vec{T} \times \hat{k} = \vec{n}$$

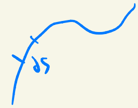
clockwise CW = inward normal

For us: will be convenient to choose CCW parametrization of outward flux to be positive.

This is a little clumsy. Let's derive a more useful 2D expression:

$$\vec{n} = \vec{T} \times \hat{k} = \left(\frac{dx}{ds}\hat{i} + \frac{dy}{ds}\hat{j} \right) \times \hat{k}$$

ds is little change in arclength



$$\vec{T} \times \hat{k} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{dx}{ds} & \frac{dy}{ds} & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \frac{dy}{ds}\hat{i} - \frac{dx}{ds}\hat{j}$$

If $\vec{F}(x,y) = M(x,y)\hat{i} + N(x,y)\hat{j}$, then

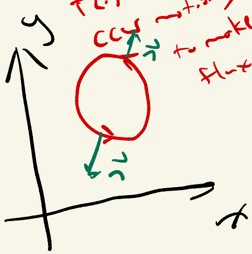
$$\vec{F} \cdot \vec{n} = \vec{F} \cdot (\vec{T} \times \hat{k})$$

$$= M(x,y)\frac{dy}{ds} - N(x,y)\frac{dx}{ds}$$

$$\oint_C \vec{F} \cdot \vec{n}\, ds = \oint_C \left(M\frac{dy}{ds} - N\frac{dx}{ds} \right) ds$$

This assumes CCW motion to make outward flux positive

$$= \oint_C M\,dy - N\,dx$$



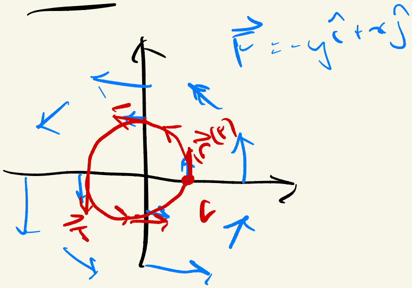
$\Rightarrow$ If C oriented CCW, the outward flux of $\vec{F} = M\hat{i} + N\hat{j}$ can be calculated by

$$\boxed{\text{Flux of } \vec{F} \text{ across } C = \oint_C M\,dy - N\,dx}$$

Ex:

Find the Flux of $\vec{F} = -y\hat{i} + x\hat{j}$ across the circle $x^2+y^2=1$ in xy plane.

Soln:



Need to orient C to go CCW. Can do by parametrization:

$$\vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j}$$

$$0 \le t \le 2\pi$$

does the trick.

$$\vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j}$$

$$\Rightarrow \quad M = -y = -\sin t$$

$$N = x = \cos t$$

$$dy = d(\sin t) = \cos t\,dt$$

$$dx = d(\cos t) = -\sin t\,dt$$

$$\text{Flux} = \oint_C M\,dy - N\,dx$$

$$= \int_0^{2\pi} -\sin t(\cos t\,dt) - \cos t(-\sin t\,dt)$$

$$= \int_0^{2\pi} -\sin t\cos t\,dt + \sin t\cos t\,dt$$

$$= 0$$

16.3: Path independence

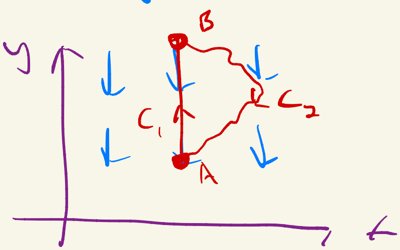
Conservation of energy in physics:

"In a closed system, the total energy is fixed over time"

Work-Energy correspondence: work "moves" energy around

Ex: Lifting and lowering a box

$\vec{F} = -\hat{j}$ Gravity field



work along C changes potential energy of the box.

From physics: potential energy of box depends only upon initial and final heights. (coordinate)

A B

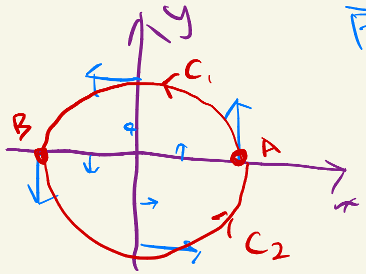
so
$$\int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r} = 0$$

in fact, any closed curve should have line integral 0!

line integral has path independence.

$\vec{F}$ will be called conservative (always conservation of energy)

Nonexample: whirlpool



$\vec{F} = -y\hat{i} + x\hat{j}$

we've seen (earlier calculation):

$$\int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r} = 2\pi$$

Not same energy as initial state A!

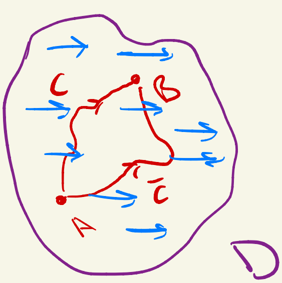
So coordinates (x, y) do not have "associated energies" (in other words, just knowing initial and final coordinates is not enough to know change in energy)

path dependent.

Def: Path independent, conservative

Let $\vec{F}$ be a continuous vector field defined on an open region D in space.

If for every pair of points A and B in D, and if every pair of paths $C, \bar{C}$ between A and B has the same line integral



$$\int_C \vec{F} \cdot d\vec{r} = \int_{\bar{C}} \vec{F} \cdot d\vec{r}$$

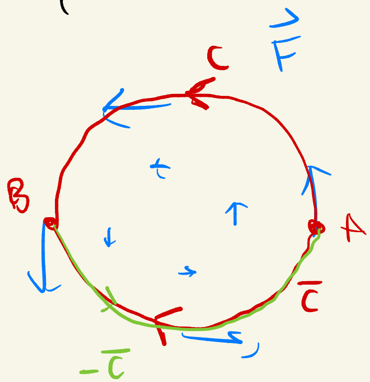
Then we say the integral $\int_C \vec{F} \cdot d\vec{r}$ is path independent and $\vec{F}$ is conservative.

Idea behind the name

If $\vec{F}$ was not conservative (e.g. whirlpool example),

then there is a pair of points A and B and paths

$C, \bar{C}$ with



$$\int_C \vec{F}\cdot d\vec{r} \neq \int_{\bar{C}} \vec{F}\cdot d\vec{r}$$

Then we can choose the bigger one

so

$$\int_C \vec{F}\cdot d\vec{r} > \int_{\bar{C}} \vec{F}\cdot d\vec{r}$$

So we can generate "infinite energy" by

$$\int_C \vec{F}\cdot d\vec{r} + \int_{-\bar{C}} \vec{F}\cdot d\vec{r} = \int_C \vec{F}\cdot d\vec{r} - \int_{\bar{C}} \vec{F}\cdot d\vec{r}$$

$$> 0$$

gained some energy!

can keep repeating to get infinite energy!

- Logical Note:

Checking conservative property requires some cleverness:

we need to be able to show for every pair A, B and all paths $C, \tilde{C}$ that the line integrals are the same. That's a lot of paths and points!

- Not enough to show it works for one pair of paths.
- If we want to show a vector field not conservative, a single pair and single path will do the trick.

- For gravity (and other conservative fields) to work, we need to have some way of consistently assigning each point to an energy.

In some sense, can we make it so where potential energy only depends on coordinate (x, y, z)...